

Convergence of the Nested Multivariate Padé Approximants

Philippe Guillaume

*Département de Mathématiques, UMR MIP 5640, INSA, Complexe Scientifique de Rangueil,
31077 Toulouse Cedex, France
E-mail: guillaum@gmm.insa-tlse.fr*

Communicated by Hans Wallin

Received November 27, 1996; accepted in revised form August 18, 1997

The nested multivariate Padé approximants were recently introduced. In the case of two variables x and y , they consist in applying the Padé approximation with respect to y to the coefficients of the Padé approximation with respect to x . The principal advantage of the method is that the computation only involves univariate Padé approximation. This allows us to obtain uniform convergence where the classical multivariate Padé approximants fail to converge. © 1998 Academic Press

1. INTRODUCTION

In this paper we study the convergence of the nested multivariate Padé approximants recently introduced in [13]. In the case of two variables x and y , these approximants have the same starting point as the Padé ◦ Padé approximants introduced by C. Chaffy-Camus in [4], consisting in the computation of the Padé approximant of the function $f_y: x \mapsto f(x, y)$ with respect to the variable x . The difference lies in the second step. Instead of computing the Padé approximant of the result with respect to the variable y , one computes the Padé approximants of the coefficients of the first step result. If y itself is a multivariable, then the algorithm is applied recursively until a single variable is obtained, which explains the term “nested” multivariate Padé approximation. The principal advantage of this method is that the algorithm only uses univariate Padé approximation. It follows that convergence results can be obtained where the classical multivariate Padé approximants fail to converge (see Remark 3.2 for some comments). We refer the reader to [13] for other properties of the nested multivariate Padé approximants. Throughout this paper, we will restrict our attention to two complex variables x and y for simplicity. However, the results apply also for more than two variables.

The outline of this paper is as follows. In Section 2, we recall the definition and the construction of the nested Padé approximants. In Section 3, we prove

the convergence of the nested Padé approximants and illustrate it on a simple example.

2. THE NESTED MULTIVARIATE PADÉ APPROXIMANTS

2.1. *Notation.* There are several definitions of a univariate Padé approximation. We use the following one [2], which includes the normalization of the denominator. Consider a power series $u(x) = \sum_{i \geq 0} u_i x^i$ and the polynomials of the form $p(x) = \sum_{i=0}^m p_i x^i$ and $q(x) = 1 + \sum_{i=1}^n q_i x^i$. If the following linear system

$$q(x) u(x) - p(x) = O(x^{m+n+1}), \quad q(0) = 1, \quad (1)$$

has a unique solution, then the fraction p/q is irreducible and is called the $[m, n]$ Padé approximant of the function u . This fraction is denoted by $[m, n]_u$. The Hankel matrix corresponding to this system (cf., e.g., [2, 11]) is denoted by $H(u, m, n)$, and the right member by $C(u, m, n)$,

$$H(u, m, n) = \begin{pmatrix} u_{m-n+1} & \cdots & u_m \\ \vdots & & \vdots \\ u_m & \cdots & u_{m+n-1} \end{pmatrix},$$

$$C(u, m, n) = - \begin{pmatrix} u_{m+1} \\ \vdots \\ u_{m+n} \end{pmatrix},$$

where $u_i := 0$ if $i < 0$. The coefficients $S = (q_n, \dots, q_1)^T$ are a solution to the system

$$H(u, m, n) S = C(u, m, n). \quad (2)$$

Equation (1) has a unique solution if and only if the determinant of $H(u, m, n)$ is non-zero. The coefficients of the polynomial $p(x)$ are obtained by taking the $m+1$ first coefficients of the Taylor series expansion at $x=0$ of the product $q(x)u(x)$.

2.2. **DEFINITION.** Let a function f be meromorphic on a polydisc $B(0, \rho_1, \rho_2) = \{(x, y) \in \mathbb{C}^2; |x| < \rho_1, |y| < \rho_2\}$ and holomorphic around $(0, 0)$ with a power series expansion

$$f(x, y) = \sum_{i, j \geq 0} f_{ij} x^i y^j.$$

Consider a fraction $R \in \mathbb{C}(y)(x)$ of the form

$$R(x, y) = \frac{P(x, y)}{Q(x, y)} = \frac{\sum_{i=0}^m r_i(y) x^i}{1 + \sum_{i=1}^n r_{m+i}(y) x^i}, \tag{3}$$

where the $r_i(y)$ are also fractions,

$$r_i(y) = \frac{p_i(y)}{q_i(y)} = \frac{\sum_{j=0}^{m_i} p_{ij} y^j}{1 + \sum_{j=1}^{n_i} q_{ij} y^j}, \quad 0 \leq i \leq n + m, \tag{4}$$

with

$$m + n = M, \quad m_i + n_i = N, \quad 0 \leq i \leq m + n, \tag{5}$$

$$\deg Q(x, 0) = n, \quad \deg q_i = n_i, \quad 0 \leq i \leq n + m. \tag{6}$$

Let $E(M, N) = \{0, 1, \dots, M\} \times \{0, 1, \dots, N\}$. For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, we denote by ∂^α the usual differential operator $\partial^\alpha = \partial^{|\alpha|} / \partial x^{\alpha_1} \partial y^{\alpha_2}$ with $|\alpha| = \alpha_1 + \alpha_2$.

DEFINITION 2.1. Consider the equation

$$\partial^\alpha R(0, 0) = \partial^\alpha f(0, 0) \quad \forall \alpha \in E(M, N). \tag{7}$$

If the fraction R is the unique solution to this equation, it is called the nested Padé approximant of order $[m, n, (m_i), (n_i)]$ of the function f , and it is denoted by $[m, n, (m_i), (n_i), x, y]_f$.

The problem of the existence of solutions to (7) is closely related to the univariate case. In the latter, it can happen that (1) has no solution (without the normalization condition $q(0) = 1$, (1) always has a non-trivial solution, but there is no guarantee that $u(x) - p(x)/q(x) = O(x^{m+n+1})$, which is the usual purpose of a Padé approximation). Similarly, it may happen that (7) has no solution. However, in the univariate case, the Montessus de Ballore theorem ensures the existence of the $[m, n]_u$ Padé approximant if n is correctly chosen and m is sufficiently large. Similarly, we will see in Theorem 3.1 that the nested Padé approximant exists as soon as the degrees of the denominators are correctly chosen and the degrees of the numerators are sufficiently large. Related to the question of existence, it could be interesting to explore a possible generalization of special series like Stieltjes series or Pólya frequency series, for which the Padé approximants always exist (see, e.g., [3]).

Concerning the uniqueness of solutions to (7), we recall the following result [13].

DEFINITION 2.2. The fraction R is said to be irreducible if the fractions $x \mapsto R(x, 0)$ and r_i , $0 \leq i \leq n + m$, are irreducible.

PROPOSITION 2.1. If the fraction R is a solution to (7) and is irreducible, then R is the unique solution to (7).

2.3. Computation. Equation (7) is a nonlinear system of $(M + 1)(N + 1)$ equations, with the same number of unknowns. However, the solution of this system is obtained in two steps by solving small linear systems.

2.3.1. First Step. Let $\mathcal{Y} \subset B(0, \rho_2)$ be an open subset where the function $y \mapsto f(0, y)$ is holomorphic and the determinant of $H(f_y, m, n)$ is non-zero, and suppose that $0 \in \mathcal{Y}$. For fixed $y \in \mathcal{Y}$, compute the $[m, n]$ Padé approximant of the function $f_y: x \mapsto f(x, y)$,

$$[m, n]_{f_y}(x) = \frac{U(x, y)}{V(x, y)} = \frac{\sum_{i=0}^m s_i(y) x^i}{1 + \sum_{i=1}^n s_{m+i}(y) x^i}, \quad (8)$$

where the vector $S(y) = (s_{m+n}(y), \dots, s_{m+1}(y))^T$ is the unique solution to the linear system

$$H(f_y, m, n) S(y) = C(f_y, m, n), \quad (9)$$

and $d^k U/dx^k(0, y) = d^k(fV)/dx^k(0, y)$, $k = 0, \dots, m$. The vector-valued function $S(y)$ is holomorphic around zero and has a power series expansion

$$S(y) = \sum_{j \geq 0} S_j y^j, \quad S_j \in \mathbb{C}^n.$$

We have also

$$H(f_y, m, n) = \sum_{j \geq 0} H_j y^j, \quad H_j \in \mathbb{C}^{n \times n},$$

$$C(f_y, m, n) = \sum_{j \geq 0} C_j y^j, \quad C_j \in \mathbb{C}^n.$$

it follows from (9) that the vectors S_j , $0 \leq j \leq N$, are solution to the systems

$$H_0 S_0 = C_0,$$

$$H_0 S_j = - \sum_{k=1}^j H_k S_{j-k} + C_j, \quad 1 \leq j \leq N,$$

which all have the *same matrix*. Their solution can be computed by using standard algorithms for univariate Padé approximants [1, 12]. The Taylor expansion of degree N of the other coefficients $s_i(y)$, $0 \leq i \leq m$, are obtained by considering the product fV .

2.3.2. *Second Step.* For $0 \leq i \leq M$, degrees m_i and n_i are chosen in such a way that $m_i + n_i = N$ (see, e.g., [9, 10] for the choice of the degrees). We suppose here that the following Padé approximants

$$r_i(y) = [m_i, n_i]_{s_i}, \quad 0 \leq i \leq M,$$

exist in the sense of definition (1), that their denominators are of degree n_i , and that $r_{m+n}(0) \neq 0$. Let

$$R(x, y) = \frac{\sum_{i=0}^m r_i(y) x^i}{1 + \sum_{i=1}^n r_{m+i}(y) x^i}.$$

We recall the following result [13].

PROPOSITION 2.2. *If the fraction R is irreducible in the sense of Definition 2.2, then R is the nested Padé approximant of order $[m, n, (m_i), (n_i)]$ of the function f .*

3. CONVERGENCE OF THE NESTED PADÉ APPROXIMANTS

The convergence result obeys the basic construction of the nested Padé approximants. It is obtained in two steps, and it is a direct consequence of the theory developed in the univariate case. It is more general than the multivariate convergence theorem [6], in the sense that:

- the coefficients f_{ij} can be used on a rectangular set of indices,
- the singular set of the function f does not need to be algebraic, that is, it may not coincide with the zeros of a polynomial in the two variables x and y ,
- it does not introduce a high-order singularity in the neighborhood of the origin.

Suppose that the function f is of the form

$$f(x, y) = \frac{u(x, y)}{v(x, y)},$$

where the functions u and v are holomorphic on the polydisc $B(0, \rho_1, \rho_2)$, and $v(x, y) = \sum_{i=0}^n v_i(y) x^i$ is a polynomial in x such that $x \mapsto v(x, 0)$ has n simple roots with $v(0, 0) \neq 0$. A particular case is when v is a polynomial in the two variables x and y . In the general case, the set where $v(x, y)$ vanishes is not necessarily algebraic.

Let $\mathcal{Y} \subset B(0, \rho_2)$ be an open subset with $0 \in \mathcal{Y}$ such that for all $y \in \mathcal{Y}$:

- $v_0(y) \neq 0, v_n(y) \neq 0$,
- the polynomial $x \mapsto v(x, y)$ has n simple roots $\alpha_i(y), 1 \leq i \leq n$, $|\alpha_i(y)| < \rho_1$, the functions α_i being holomorphic on \mathcal{Y} (simple roots can be replaced by roots of constant multiplicity),
- $u(x, y) \neq 0$ if $v(x, y) = 0$.

Let \mathcal{O} be the open subset

$$\mathcal{O} = \{(x, y) \in B(0, \rho_1, \rho_2), y \in \mathcal{Y}, v(x, y) \neq 0\}.$$

In the intermediate Padé approximant $[m, n]_{f_y}(x) = U_m(x, y)/V_m(x, y)$ (8), whose existence will be proved for m sufficiently large in Lemma 3.2, we add the subscript m to indicate the dependence on m (n is fixed):

$$U_m(x, y) = \sum_{i=0}^m s_i^m(y) x^i, \quad V_m(x, y) = 1 + \sum_{i=1}^n s_{m+i}^m(y) x^i.$$

THEOREM 3.1. *The sequence of nested Padé approximants converges uniformly to f on all compact subsets of \mathcal{O} in the following sense: for all $\varepsilon > 0$ and all compact subsets $\mathcal{K} \subset \mathcal{O}$, there is an integer L such that for all $m \geq L$, there exist integers N_m and $n_i^m, 0 \leq i \leq m+n$, such that for all $N \geq N_m$, the nested Padé approximant $[m, n, (N - n_i^m), (n_i^m), x, y]_f$ of the function f is well defined and*

$$\sup_{(x, y) \in \mathcal{K}} |f(x, y) - [m, n, (N - n_i^m), (n_i^m), x, y]_f(x, y)| < \varepsilon.$$

Each n_i^m can be chosen equal to the number of poles (counted with multiplicity) within the ball $B(0, \rho_2)$ of the function $s_i^m, 0 \leq i \leq m+n$.

Remark 3.1. The assumption on the degrees of the denominators are very close to the assumptions of the classical univariate Montessus de Ballore theorem [2, 3]. Although the number of poles of the function s_i^m is not known, the technique described in [10] for counting the number of poles of meromorphic functions within a ball can be used here. A difficulty which could appear for a practical use is that the numbers n_i^m may increase

with m . However, numerical tests have not shown such a growth, although we have not been able to prove that these numbers remain bounded. The problem of the existence of an upper bound of the numbers n_i^m remains open.

Proof. We need some notation and the following lemma. Define

$$\tilde{U}_m(x, y) = \frac{U_m(x, y)}{s_{m+n}^m(y)}, \quad \tilde{V}_m(x, y) = \frac{V_m(x, y)}{s_{m+n}^m(y)}.$$

After dividing in f the numerator and the denominator by the function v_n (which does not vanish on \mathcal{Y}), the function f can be put in the following form which fits the form \tilde{U}_m/\tilde{V}_m of $[m, n]_{f_y}$,

$$f(x, y) = \frac{h(x, y)}{g(x, y)}, \quad g(x, y) = \sum_{i=0}^{n-1} g_i(y) x^i + x^n,$$

where the functions h and g are meromorphic on $B(0, \rho_1, \rho_2)$ and holomorphic on $B(0, \rho_1) \times \mathcal{Y}$.

LEMMA 3.2. *for all compact subsets $\mathcal{K}_y \subset \mathcal{Y}$, there is an integer L such that for all $m \geq L$ and all $y \in \mathcal{K}_y$, there is a unique intermediate Padé approximant $[m, n]_{f_y}$. The sequence $([m, n]_{f_y})_{m \geq L}$ converges uniformly to f on all compact subsets of $(B(0, \rho_1) \times \mathcal{K}_y) \cap \mathcal{O}$.*

We give the main line of the proof which is quite similar to the proof of Lemma 1 in [4] and is based on Saff's technique for proving the Montessus de Ballore theorem [16]. The key point is to put $\tilde{V}_m(x, y)$ in the form

$$\tilde{V}_m(x, y) = g(x, y) + \sum_{k=0}^{n-1} w_k^m(y) W_k(x, y),$$

where $W_0 \equiv 1$, $W_k(x, y) = (x - \alpha_1(y)) \cdots (x - \alpha_k(y))$ is a polynomial in x of degree k , holomorphic on $\mathbb{C} \times \mathcal{Y}$, and to reformulate the problem as follows.

For fixed $y \in \mathcal{Y}$, let $\pi_m(x, y)$ be the Taylor expansion of degree $m + n$ at $x = 0$ of the function $x \mapsto \tilde{V}_m(x, y) h(x, y)$. The polynomial in x , $\tilde{V}_m(x, y)$ is chosen in such a way that the polynomial in x , $\pi_m(x, y)$ vanishes at the n roots $\alpha_k(y)$ of $g(\cdot, y)$, that is, there exists a polynomial in x , $\tilde{U}_m(x, y)$ such that $\pi_m(x, y) = \tilde{U}_m(x, y) g(x, y)$. If $\tilde{V}_m(0, y) \neq 0$, these conditions coincide with the conditions defining U_m and V_m .

Without any loss of generality, we can assume that the functions $x \mapsto u(x, y)$ and $x \mapsto v(x, y)$ are holomorphic on a neighborhood of $\bar{B}(0, \rho_1)$, independent of $y \in \mathcal{Y}$. Thanks to Hermite's formula

$$\pi_m(x, y) = \frac{1}{2i\pi} \int_{|z|=\rho_1} \left(1 - \left(\frac{x}{z}\right)^{m+n+1}\right) \frac{\tilde{V}_m(z, y) h(z, y)}{z-x} dz, \quad (10)$$

the coefficients $w_0^m(y), \dots, w_{n-1}^m(y)$ are a solution to the system

$$\sum_{k=0}^{n-1} A_{jk}^m(y) w_k^m(y) = B_j^m(y), \quad j = 1, 2, \dots, n,$$

$$A_{jk}^m(y) = \frac{1}{2i\pi} \int_{|z|=\rho_1} \left(1 - \left(\frac{\alpha_j(y)}{z}\right)^{m+n+1}\right) \frac{W_k(z, y) h(z, y)}{z - \alpha_j(y)} dz,$$

$$B_j^m(y) = \frac{1}{2i\pi} \int_{|z|=\rho_1} \left(\frac{\alpha_j(y)}{z}\right)^{m+n+1} \frac{g(z, y) h(z, y)}{z - \alpha_j(y)} dz,$$

which converges uniformly on \mathcal{K}_y to a triangular and homogeneous system. Due to $u(x, y) \neq 0$ if $v(x, y) = 0$, the diagonal elements are non-zero and this system is invertible. Thus, for $m \geq L$ sufficiently large and $y \in \mathcal{K}_y$, the coefficients $w_0^m(y), \dots, w_{n-1}^m(y)$ are uniquely determined, holomorphic in y , they converge uniformly to zero, and \tilde{V}_m converges uniformly to g on all compact subsets of $\mathbb{C} \times \mathcal{K}_y$.

Using Eq. (10) again, we can bound $|\pi_m(x, y) - \tilde{V}_m(x, y) h(x, y)|$ by

$$\frac{|x/\rho_1|^{m+n+1}}{1 - |x/\rho_1|} \sup_{|z|=\rho_1, t \in \mathcal{K}_y} |\tilde{V}_m(z, t) h(z, t)|.$$

Hence $\pi_m = \tilde{U}_m g$ converges to gh uniformly on all compact subsets of $B(0, \rho_1) \times \mathcal{K}_y$. The division by g gives the uniform convergence of \tilde{U}_m to h on all compact subsets of $(B(0, \rho_1) \times \mathcal{K}_y) \cap \mathcal{O}$, and finally the division by \tilde{V}_m gives the uniform convergence of \tilde{U}_m/\tilde{V}_m to f on all compact subsets of $(B(0, \rho_1) \times \mathcal{K}_y) \cap \mathcal{O}$. Moreover, as $g(0, y) \neq 0$ for $y \in \mathcal{Y}$, we have $\tilde{V}_m(0, y) \neq 0$ for m sufficiently large, and $U_m/V_m(\cdot, y) = \tilde{U}_m/\tilde{V}_m(\cdot, y) = [m, n]_f$ for all $y \in \mathcal{K}_y$.

Proof of the Theorem. We have to check that the functions $s_i^m(y)$, $0 \leq i \leq m+n$, are holomorphic around zero, so that their Padé approximant can be defined, and to choose the degrees of the denominators in $[m_i, n_i]_{s_i^m}$. Then we apply the Montessus de Ballore theorem.

Without any loss of generality, we can assume that the interior of \mathcal{H} contains the origin. It follows from the lemma that for m sufficiently large, V_m is well defined and holomorphic around the origin, thus the functions $s_i^m(y)$, $m + 1 \leq i \leq m + n$ are holomorphic around zero. The function U_m is then also holomorphic around the origin, thus the functions $s_i^m(y)$, $0 \leq i \leq m$, are holomorphic around the origin. Due to the construction of the intermediate Padé approximant (see (9)) and to the assumptions made on f , the functions s_i^m are meromorphic on $B(0, \rho_2)$, and they have a finite number n_i^m of poles (counted with multiplicity) within this ball. Thanks to the Montessus de Ballore theorem, there are integers L_i^m , $0 \leq i \leq m + n$, such that $[m_i, n_i^m]_{s_i^m}$ is well defined for $m_i \geq L_i^m$, and each sequence $([m_i, n_i^m]_{s_i^m})_{m_i \geq L_i^m}$, $0 \leq i \leq m + n$, converges to s_i^m uniformly on all compact subsets of \mathcal{D} . It follows from Proposition 2.1 that the nested Padé approximant $[m, n(N - n_i^m), (n_i^m), x, y]_f$ is well defined for $N \geq \max_{0 \leq i \leq m+n} L_i^m + n_i^m$, and the proof of the convergence is then easy to achieve. ■

Remark 3.2. One does not seem to know today in which cases the general multivariate Padé approximants converge uniformly on compact subsets. related to this question, several attempts have been made for generalizing the Montessus de Ballore theorem to more than one variable. The problem is difficult: an error in the proof of Theorems 3 to 6 in [5] has been observed by Karlsson and Wallin in [14], an error in the proof of Theorem 3.1 in [15] has been observed by Cuyt in [6], there is an error in Eq. (9) of the proof of Theorem 2 in [7], a counterexample to the theorem is $f(x, y) = 1/(1 - x)(1 - y)$, and the same error appears in the proof of Theorem 1 in [8]. All these errors come from a wrong generalization of the formula $d/dx^k(g(Qf - P))(x) = d/dx^k(gQf)(x)$ for $k > m + n$ (it is assumed here that $P/Q = [m, n]_f$, $f = h/g$, h holomorphic and g polynomial of degree n).

Apparently, uniform convergence has only been proved in two cases: the case of the homogeneous Padé approximants [6, 14], and the case of the Padé ◦ Padé approximants [4].

3.1 EXAMPLE. Consider the function:

$$f(x, y) = \left(\frac{4}{4 - x^2 - y^2} + \frac{5}{5 - x - 5y} \right) \exp \left(\frac{xy}{2} \right).$$

The nested Padé approximants $[m, n, (m_i), (n_i), y, x]_f$ are denoted here by $R_m(x, y)$, and computed for $n = 3$ and $4 \leq m \leq 20$. The exchange of x and y has been done for graphical convenience. the degrees m_i and n_i are

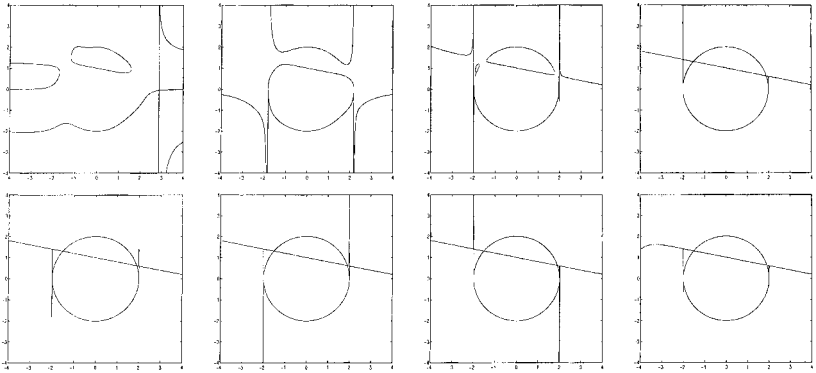


FIG. 1. Real zeros of the denominator for $m=4, 6, 8, 10, 12, 14, 16, 20$.

determined automatically [9, 10] in the second step consisting of univariate Padé approximations.

Figure 1 shows the singular set of the nested approximants for some values of m . The last picture shows the numerical instability which appears in double precision for large values of m , since the condition number of the matrix H_0 used at the first step increases with m (see Table I).

Table I compares the approximations $R_m(x, y)$ and $f(x, y)$ for different values of m and (x, y) , and the last line gives the condition number of the matrix H_0 obtained in the first step.

Figure 2 shows the convergence of the approximants. Each graphic gives the values of the function f (dashed line), and of an approximant (solid line) computed on the diagonal segment $\{(x, y) \in \mathbb{R}^2; y = x, -3 \leq x \leq 3\}$ (other families of points have been tested and have given similar results). Two extra singularities appear at $x = -2$ and $x = 2$ where the intermediate Padé approximants $[m, n]_{f_x}$ are not defined.

TABLE I

Convergence and Condition Number

	$m=4$	$m=8$	$m=12$	$m=14$	$m=16$	$m=20$	$f(x, y)$
$x = y = 1$	-4.9613	-4.7727	-4.9462	-4.9462	-4.9462	-4.9462	-4.9462
$x = y = 1.7$	-8.799	-6.596	-13.609	-13.610	-13.610	-13.604	-13.611
$x = y = 2.5$	-14.96	16.63	-21.58	-22.47	-22.52	-18.11	-22.09
$\text{cond}(H_0)$	$1.7 \cdot 10^2$	$2.6 \cdot 10^3$	$4.1 \cdot 10^4$	$1.7 \cdot 10^5$	$6.7 \cdot 10^5$	$1.0 \cdot 10^7$	

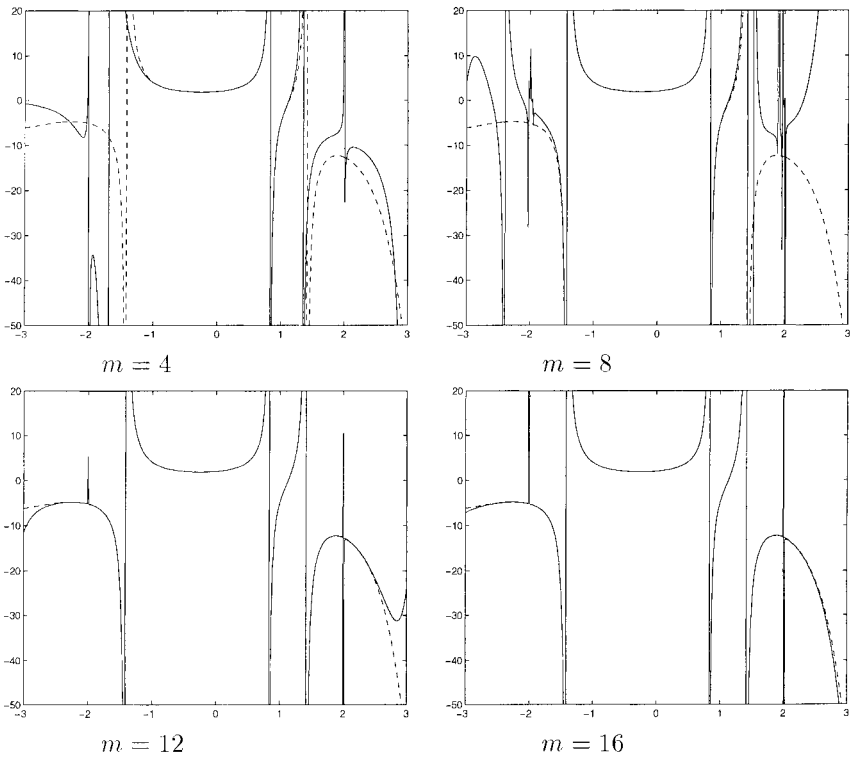


FIG. 2. Convergence of $R_m(x, y)$ to $f(x, y)$.

4. CONCLUSION

We have shown the convergence of the nested multivariate Padé approximants, whose practical interest is the reduction to the univariate Padé approximation. Some open questions remain, like the increasing of the degrees n_i^m of the denominators, or a characterization of the series for which the nested Padé approximants exist for all degrees.

ACKNOWLEDGMENT

I am grateful to the referees whose comments have helped me in improving the presentation of this paper and correcting some errors in the first document.

REFERENCES

1. G. A. Baker, Recursive calculation of Padé approximants, in "Padé Approximants and Their Applications" (P. R. Graves-Morris, Ed.), pp. 83–91, Academic Press, London, 1973.
2. C. Brezinski, Approximants de Padé, in "Analyse Numérique," pp. 416–458, Collection Enseignement des Sciences, Hermann, Paris, 1991.
3. C. Brezinski and J. van Iseghem, Padé approximations, in "Handbook of Numerical Analysis" (P. G. Ciarlet and J. L. Lions, Eds.), North-Holland, Amsterdam, 1992.
4. C. Chaffy-Camus, Convergence uniforme d'une nouvelle classe d'approximants de Padé à plusieurs variables, *C. R. Acad. Sci. Paris. Sér. I* **306** (1988), 387–392.
5. J. S. R. Chisholm and P. R. Graves-Morris, Generalizations of the theorem of de Montessus to two-variable approximants, *Proc. Roy. Soc. London Ser. A* **342** (1975), 341–372.
6. A. Cuyt, A Montessus de Ballore theorem for multivariate Padé approximants, *J. Approx. Theory* **43** (1985), 43–52.
7. A. Cuyt, A multivariate convergence theorem of the "de Montessus de Ballore" type, *J. Comput. Appl. Math.* **32** (1990), 47–57.
8. A. Cuyt, Extension of "A multivariate convergence theorem of the "de Montessus de Ballore" type to multipoles," *J. Comput. Appl. Math.* **41** (1992), 323–330.
9. J. Gilewicz, M. Griffiths, and A. Mortada, Modélisation des connaissances servant comme base d'un système expert d'aide au choix du meilleur approximant de Padé, *AF CET/ INTERFACES* **81** (1989), 4–11.
10. B. Gleyse and V. Kaliaguine, On algebraic computation of number of poles of meromorphic functions in the unit disk, in "Nonlinear Numerical Methods and Rational Approximation, II" (A. Cuyt, Ed.), pp. 241–246, Kluwer Academic, Netherlands, 1994.
11. W. B. Gragg, The Padé table and its relation to certain algorithms of numerical analysis, *SIAM Rev.* **14** (1972), 1–62.
12. P. R. Graves-Morris, The numerical calculation of Padé approximants, in "Lecture Notes in Mathematics," Vol. 765, pp. 231–245, Springer-Verlag, New York/Berlin, 1979.
13. Ph. Guillaume, Nested multivariate Padé approximants, *J. Comput. Appl. Math.* **81** (1997).
14. J. Karlsson and H. Wallin, Rational approximation by an interpolation procedure in several variables, in "Padé and Rational Approximation: Theory and Applications" (E. Saaf and R. Varga, Eds.), pp. 83–100, Academic Press, New York, 1977.
15. C. H. Lutterodt, On a partial converse of the Montessus de Ballore theorem in \mathbb{C}^n , *J. Math. Anal. Appl.* **40** (1984), 216–225.
16. E. B. Saaf, An extension of Montessus de Ballore's theorem on the convergence of interpolating rational functions, *J. Approx. Theory* **6** (1972), 63–67.